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Abstract

A new theory of homogeneity for neutral type systems with application to fast stabilization of the 2nd-order integrator is proposed. It is assumed that only the position is available for measurements, and the designed feedback uses the output and its delayed values without an estimation of velocity. It is shown that by selecting the closed-loop system to be homogeneous with negative or positive degree, it is possible to accelerate the rate of convergence in the system at the price of a small steady-state error. Robustness of the developed stabilization strategy with respect to exogenous perturbations is investigated. The efficiency of the proposed control is demonstrated in simulations.

1 Introduction

The design of regulators for dynamical systems is a fundamental and complex problem. Important features of different existing methods for control synthesis are the achievable quality of transients and robustness against exogenous perturbations and noises. Very frequently the design methods are oriented on various canonical models, and the linear ones are the most popular. Then a system of second order presents a conventional benchmarking tool. If non-asymptotic rates of convergence (i.e. finite-time or fixed-time [1]) are needed in the closed-loop system, then usually homogeneous systems come to the attention as canonical dynamics.

The theory of homogeneous dynamical systems is well-developed for continuous time-invariant differential equations [2, 3, 4, 5], time-varying dynamics [6, 7] or time-delay systems [8, 9, 10, 11, 12, 13]). The main feature of a homogeneous system (described by ordinary differential equation) is that its local behavior of trajectories is the same as global one [1], while for time-delay homogeneous systems the delay-independent (DI) stability follows [9]. The rate of convergence for homogeneous ordinary differential equations is related with degree of homogeneity [1], but for time-delay systems the links are not so straightforward [14]. In addition, the homogeneous stable/unstable systems (described again by ordinary differential equations) admit homogeneous Lyapunov functions [5, 15, 16].

From another side, influence of a delay on the system stability is vital in many cases [17, 18]. Despite of variety of applications, most of them deal with the linear time-delay models, which is originated by complexity of stability analysis for (nonlinear) time-delay systems [18]. However, in some cases introduction of a delay may lead to an improvement of the system performance (see [19, 20]). The idea of these papers is that unmeasured components of the state can be calculated using delayed values of the measured variables.

The goal of this work is to extend the results obtained in [19, 20] for linear systems to a nonlinear homogeneous scenario restricting for brevity the attention to the case of a second order model. Since the stability analysis in [19, 20] is based on Lyapunov-Krasovskii functionals that depend on state derivatives, we start with an extension of the results of [8, 9] to neutral type systems. Second, a design method is proposed, which uses the position and its delayed values for practical output stabilization with accelerated convergence rates (as it is mentioned above, the finite-time stability concept for time-delay systems is not so natural as for the ordinary differential equations [14], then the convergence faster than any exponential has been found more appropriate [21]). Robustness with respect to external disturbances is analyzed. The preliminary studies were per-

⋆ This work was partially supported by the Government of Russian Federation (Grant 08-08), by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898, and by Israel Science Foundation (grant No 673/19).

formed in [22] (without proofs, extension of homogeneity notion to neutral type systems and analysis of robustness).

2 Preliminaries

Let $\mathbb{R}$ be the set of reals and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$.

For a Lebesgue measurable function of time $d : [a, b] \to \mathbb{R}^m$, $[a, b] \subset \mathbb{R}$, define the norm $\|d\|_{[a,b)} = \text{ess sup}_{t\in[a,b]}|d(t)|$, where $|\cdot|$ is the standard Euclidean norm in $\mathbb{R}^m$, then $||d||_\infty = \|d\|_{[0,+\infty)}$ and the space of d with $\|d\|_{[a,b)} < +\infty$ ($||d||_\infty < +\infty$) we further denote as $\mathcal{L}^m_{[a,b]}$ ($\mathcal{L}^m_\infty$). Denote by $C^n_{[a,b]}$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_{[a,b)} = \sup_{a\leq s\leq b} |\phi(s)|$; and by $\mathbb{W}^{1,\infty}_{[a,b]}$ the Sobolev space of absolutely continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the norm $\|\phi\|_\mathbb{W} = \|\phi\|_{[a,b)} + ||\dot\phi||_{[a,b)} < +\infty$ [1], where $\dot\phi(s) = \frac{\partial\phi(s)}{\partial s}$ is a Lebesgue measurable essentially bounded function, i.e. $\dot\phi \in \mathcal{L}^n_{[a,b]}$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is a decreasing to zero for any fixed $r > 0$.

The symbol $\overline{1, m}$ is used to denote a sequence of integers $1, ..., m$. The minimum and maximum eigenvalues of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ are denoted as $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively.

2.1 Neutral time-delay systems

Consider an autonomous functional differential equation of neutral type with inputs [23]:

$$\dot x(t) = f(x_t, \dot x_t, d(t)) \tag{1}$$

for almost all $t \geq 0$, where $x(t) \in \mathbb{R}^n$ and $x_t \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ is the state function, $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$, with $\dot x_t \in \mathcal{L}^n_{[-\tau,0]}$; $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}^m_\infty$; $f : \mathbb{W}^{1,\infty}_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function ensuring forward uniqueness and existence of the system solutions [23], $f(0, 0, 0) = 0$. For the initial function $x_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and disturbance $d \in \mathcal{L}^m_\infty$ denote a unique solution of (1) by $x(t, x_0, d)$, which is an absolutely continuous function defined on some maximal time interval $[-\tau, T)$ for $T > 0$, then $x_t(x_0, d) \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ represents the corresponding state function with $x_t(s, x_0, d) = x(t + s, x_0, d)$ for $-\tau \leq s \leq 0$.

[1] In [23, 24] another norm for the state space of time-delay systems has been used: $\|\phi\|_\mathbb{W} = \|\phi\|_{[a,b)} + \sqrt{\int_a^b |\dot\phi(s)|^2 ds}$. The norm $|| \cdot ||_{[a,b)}$ is selected for $\dot\phi$ in this work to perform an extension of homogeneity concept in Section 3.

Given a locally Lipschitz continuous functional $V : \mathbb{R} \times \mathbb{W}^{1,\infty}_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ define:

$$D^+V(t, \phi, d) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t + h, x_h(\phi, \tilde d), \dot x_h(\phi, \tilde d)) - V(t, \phi, \dot\phi)].$$

Here $x_h(\phi, \tilde d)$ is a solution of (1) with $\phi \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $\tilde d(s) = d$ for $d \in \mathbb{R}^m$ and all $s = [0, h)$.

2.2 ISS of time delay systems

The input-to-state stability (ISS) property is an extension of the conventional stability paradigm to the systems with external inputs [24, 25, 26].

Definition 1 *[24, 26] The system (1) is called practical ISS, if there exist $q \geq 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$*

$$|x(t, x_0, d)| \leq \beta(\|x_0\|_\mathbb{W}, t) + \gamma(||d||_\infty) + q \quad \forall t \geq 0.$$

If $q = 0$ then (1) is called ISS.

Definition 2 *The system (1) is said to possess the practical asymptotic gain (AG) property, if there exist $q \geq 0$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$*

$$\lim_{t\to+\infty} |x(t, x_0, d)| \leq \gamma(||d||_\infty) + q.$$

If this inequality is satisfied for $\|x_0\|_\mathbb{W} \leq X$ and $||d||_\infty \leq D$ with some $X \geq 0$ and $D \geq 0$, then such a system admits a practical (X, D)-AG property.

Definition 3 *The system (1) is said to admit the practical global stability (GS) property, if there exist $q \geq 0$ and $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for all $x_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d \in \mathcal{L}^m_\infty$*

$$|x(t, x_0, d)| \leq \max\{\sigma_1(\|x_0\|_\mathbb{W}), \sigma_2(||d||_\infty)\} + q \quad \forall t \geq 0.$$

If this inequality is satisfied for $\|x_0\|_\mathbb{W} \leq X$ and $||d||_\infty \leq D$ with some $X \geq 0$ and $D \geq 0$, then such a system possesses practical (X, D)-local stability (LS) property.

As it follows from the definitions above, a practical ISS system has practical AG and practical GS properties, and for a system in (1) described by an ordinary differential equation the converse implication also holds. For the system (1) with $d = 0$, AG and GS properties imply global asymptotic stability (GAS) in the usual sense, while $(X, 0)$-AG and $(X, 0)$-LS stay for the local one from initial conditions $\|x_0\|_\mathbb{W} \leq X$. The same properties can be defined with respect to a set by replacing the distance to the origin $\|x\|_\mathbb{W}$ with the distance to the set.

Definition 4 *A locally Lipschitz continuous functional $V : \mathbb{R} \times \mathbb{W}^{1,\infty}_{[-\tau,0]} \times \mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called simple if $D^+V(t, \phi, d)$ is independent on $\ddot\phi(t + s)$ for $-\tau \leq s \leq 0$.*

For instance, a locally Lipschitz functional $V : C^n_{[-\tau,0]} \to \mathbb{R}_+$ is simple, another example of a simple functional is given in Theorem 2 below.

Definition 5 *[24, 26] A simple $V : \mathbb{R}\times\mathbb{W}^{1,\infty}_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}_+$ is called practical ISS Lyapunov-Krasovskii functional (LKF) for the system (1) if there exist $r \geq 0$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha, \chi \in \mathcal{K}$ such that for all $t \in \mathbb{R}_+$, $\phi \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d \in \mathbb{R}^m$:*

$$\alpha_1(|\phi(0)|) \leq V(t,\phi,\dot{\phi}) \leq \alpha_2(\|\phi\|_{\mathbb{W}}),$$
$$V(t,\phi,\dot{\phi}) \geq \max\{r,\chi(|d|)\} \Longrightarrow D^+V(t,\phi,d) \leq -\alpha(V(t,\phi,\dot{\phi})).$$

If $r = 0$ then V is an ISS LKF.

Theorem 1 *[24] If there exists a (practical) ISS LKF for (1), then it is (practical) ISS with $\gamma = \alpha_1^{-1}\circ\chi$.*

3 Homogeneity

For any $r_i > 0$, $i = \overline{1,n}$ and $\lambda > 0$, define the dilation matrix $\Lambda_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$ and the vector of weights $r = [r_1, ..., r_n]^T$ (for any $\nu \in \mathbb{R}$ denote $r+\nu = [r_1+\nu, ..., r_n+\nu]^T$, $r_{\max} = \max_{1\leq i\leq n} r_i$ and $r_{\min} = \min_{1\leq i\leq n} r_i$). For the retarded type functional differential equations a concept of homogeneity has been introduced in [27] with a list of useful characterizations in [9]. Below we present an extension of those concepts and results for neutral type systems.

Definition 6 *The function $g : \mathbb{W}^{1,\infty}_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $\phi \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ the relation*

$$g(\Lambda_r(\lambda)\phi, \Lambda_{r+\nu}(\lambda)\dot{\phi}) = \lambda^\nu g(\phi,\dot{\phi})$$

holds for some $\nu \in \mathbb{R}$ and all $\lambda > 0$. The vector field $F : \mathbb{W}^{1,\infty}_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]} \to \mathbb{R}^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $\phi \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$ the relation

$$F(\Lambda_r(\lambda)\phi, \Lambda_{r+\nu}(\lambda)\dot{\phi}) = \lambda^\nu\Lambda_r(\lambda)F(\phi,\dot{\phi})$$

holds for some $\nu \geq -r_{\min}$ and all $\lambda > 0$. In both cases, the constant ν is called the degree of homogeneity.

The system (1) is called r-homogeneous of degree $\nu \geq -r_{\min}$ if $F(\phi,\dot{\phi}) = f(\phi,\dot{\phi},0)$ admits this property.

A simple scalar example is

$$F(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}(t-h) - \phi(0)^3,$$

which is homogeneous for $r = 1$ and $\nu = 2$. Indeed, in this case $\Lambda_r(\lambda) = \lambda^r = \lambda$ and for any $\lambda > 0$:

$$F(\lambda^r\phi, \lambda^{r+\nu}\dot{\phi}) = \lambda^3\left(\frac{1}{2}\dot{\phi}(t-h) - \phi(0)^3\right) = \lambda^{r+\nu}F(\phi,\dot{\phi}).$$

The introduced notion of homogeneity in $\mathbb{W}^{1,\infty}_{[-\tau,0]}$ is reduced to one defined for $C^n_{[-\tau,0]}$ if there is no dependence on $\dot{\phi}$ [27] or to the standard one in $\mathbb{R}^n$ [2, 5] under a vector argument substitution.

For any $x \in \mathbb{R}^n$ the homogeneous norm is defined as

$$|x|_r^\varrho = \sum_{i=1}^n |x_i|^{\varrho/r_i},\ \varrho \geq r_{\max}.$$

For all $x \in \mathbb{R}^n$, the norms $|x|$ is related with $|x|_r$:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \overline{\sigma}_r(|x|_r)$$

for some $\underline{\sigma}_r, \overline{\sigma}_r \in \mathcal{K}_\infty$. The homogeneous norm is a r-homogeneous function of degree one: $|\Lambda_r(\lambda)x|_r = \lambda|x|_r$ for all $x \in \mathbb{R}^n$. Similarly, for any $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$, $-\infty \leq a < b \leq +\infty$ the homogeneous norm can be defined as follows

$$||\phi||_r^\varrho = \sum_{i=1}^n ||\phi_i||_{[a,b)}^{\frac{\varrho}{r_i}} + \sum_{i=1}^n ||\dot{\phi}_i||_{[a,b)}^{\frac{\varrho}{r_i+\nu}}$$

for $\varrho \geq r_{\max} + \max\{0,\nu\}$.

Lemma 1 *There exist two functions $\underline{\rho}_r, \overline{\rho}_r \in \mathcal{K}_\infty$ such that for all $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$*

$$\underline{\rho}_r(||\phi||_r) \leq ||\phi||_{\mathbb{W}} \leq \overline{\rho}_r(||\phi||_r). \qquad (2)$$

PROOF. Define $\psi_r(s) = \begin{cases} s^{r_{\max}} & s \geq 1 \\ s^{r_{\min}} & s < 1 \end{cases}$, then

$$||\phi||_r = \left(\sum_{i=1}^n ||\phi_i||_{[a,b)}^{\frac{\varrho}{r_i}} + \sum_{i=1}^n ||\dot{\phi}_i||_{[a,b)}^{\frac{\varrho}{r_i+\nu}}\right)^{1/\varrho}$$
$$\leq n^{\frac{1}{\varrho}}\left(\psi_{\frac{\varrho}{r}}(\|\phi\|_{\mathbb{W}}) + \psi_{\frac{\varrho}{r+\nu}}(\|\phi\|_{\mathbb{W}})\right)^{1/\varrho}$$

and inverting the nonlinear function in the right-hand side we obtain an estimate for $\underline{\rho}_r$. By definition of the norm $||\cdot||_r$ the inequalities $||\phi||_r^{r_i} \geq ||\phi_i||_{[a,b)}$ and $||\phi||_r^{r_i+\nu} \geq ||\dot{\phi}_i||_{[a,b)}$ are satisfied for any $\phi \in \mathbb{W}^{1,\infty}_{[a,b]}$, then

$$\|\phi\|_{\mathbb{W}} = \sup_{a\leq s\leq b}\sqrt{\sum_{i=1}^n \phi_i^2(s)} + \sup_{a\leq s\leq b}\sqrt{\sum_{i=1}^n \dot{\phi}_i^2(s)}$$
$$\leq \sqrt{\sum_{i=1}^n ||\phi_i||_{[a,b)}^2} + \sqrt{\sum_{i=1}^n ||\dot{\phi}_i||_{[a,b)}^2}$$
$$\leq \sqrt{n}\left(\psi_r(||\phi||_r) + \psi_{r+\nu}(||\phi||_r)\right),$$

which gives an expression for the function $\overline{\rho}_r$.

Remark 1 *If the norm $\|\cdot\|_{\mathbb{W}}$ is defined as in [23, 24]:*

$$\|\phi\|_{\mathbb{W}} = \|\phi\|_{[a,b)} + \sqrt{\int_a^b |\dot{\phi}(s)|^2 ds},$$

then the property (2) is hard to establish. Indeed, in order to define a homogeneous norm, the square power under integral has to be replaced by a power related with the weights r, and boundedness of $\int_a^b |\dot{\phi}_i(s)|^2 ds$ is hard to link with the same property of $\int_a^b |\dot{\phi}_i(s)|^{\frac{\varrho}{r_i+\nu}} ds$. A solution consists in the use of $\mathbb{W}^{1,\infty}_{[a,b]}$, since if $||\dot{\phi}||_{[a,b)} < +\infty$, then all integrals above are also bounded.

The homogeneous norm in a Banach space has the same homogeneity property that is

$$||\Lambda_r(\lambda)\phi||_r=\left(\sum_{i=1}^n||\lambda^{r_i}\phi_i||_{[a,b)}^{\frac{\varrho}{r_i}}+\sum_{i=1}^n||\lambda^{r_i+\nu}\dot{\phi}_i||_{[a,b)}^{\frac{\varrho}{r_i+\nu}}\right)^{1/\varrho}=\lambda||\phi||_r$$

for all $\phi\in\mathbb{W}^{1,\infty}_{[a,b]}$. In $\mathbb{W}^{1,\infty}_{[-\tau,0]}$, for a radius $\rho>0$, denote the corresponding sphere $\mathcal{S}^\tau_\rho=\{\phi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}:||\phi||_r=\rho\}$ and the closed ball $B^\tau_\rho=\{\phi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}:||\phi||_r\le\rho\}$.

An advantage of homogeneous systems described by nonlinear ordinary differential equations is that any of their solution can be obtained from another solution under the dilation re-scaling and a suitable time parametrization. A similar property holds for neutral functional differential homogeneous systems:

Lemma 2 *Let $x(t,x_0,d)$ be a solution of the system*

$$\dot{x}(t)=f(x_t,\dot{x}_t,d(t)),\ x_t\in\mathbb{W}^{1,\infty}_{[-\tau,0]},\ d(t)\in\mathbb{R}^m \qquad (3)$$

for initial condition $x_0\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$, $\tau\in(0,+\infty)$ and input $d\in\mathcal{L}^m_\infty$, where $f:\mathbb{W}^{1,\infty}_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\times\mathbb{R}^m\to\mathbb{R}^n$ admits the following property for all $\phi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$ and $d\in\mathbb{R}^m$:

$$f(\Lambda_r(\lambda)\phi,\Lambda_{r+\nu}(\lambda)\dot{\phi},\Lambda_{\tilde{r}}(\lambda)d)=\lambda^\nu\Lambda_r(\lambda)f(\phi,\dot{\phi},d) \qquad (4)$$

for some vectors of weights $r=[r_1,...,r_n]^T$ ($r_i>0$, $i=\overline{1,n}$) and $\tilde{r}=[\tilde{r}_1,...,\tilde{r}_m]^T$ ($\tilde{r}_j\ge0$, $j=\overline{1,m}$) with some $\nu\ge-r_{\min}$. Then for any $\lambda>0$ the system

$$\dot{y}(t)=f(y_t,\dot{y}_t,\delta(t)),\ y_t\in\mathbb{W}^{1,\infty}_{[-\lambda^{-\nu}\tau,0]} \qquad (5)$$

has a solution $y(t,y_0,\delta)=\Lambda_r(\lambda)x(\lambda^\nu t,x_0,d)$ with initial condition $y_0\in\mathbb{W}^{1,\infty}_{[-\lambda^{-\nu}\tau,0]}$, $y_0(\theta)=\Lambda_r(\lambda)x_0(\lambda^\nu\theta)$ for all $\theta\in[-\lambda^{-\nu}\tau,0]$ and $\delta(t)=\Lambda_{\tilde{r}}(\lambda)d(\lambda^\nu t)$ for all $t\ge0$.

In the formulation of the proposition the same symbol f is used to define the function $f:\mathbb{W}^{1,\infty}_{[-\tau,0]}\times\mathcal{L}^n_{[-\tau,0]}\times\mathbb{R}^m\to\mathbb{R}^n$ and its counterpart obtained after scaling the delay argument $f:\mathbb{W}^{1,\infty}_{[-\lambda^{-\nu}\tau,0]}\times\mathcal{L}^n_{[-\lambda^{-\nu}\tau,0]}\times\mathbb{R}^m\to\mathbb{R}^n$.

PROOF. By definition $\frac{dx(t,x_0,d)}{dt}=f(x_t,\dot{x}_t,d(t))$, where $x_t(s)=x(t+s,x_0,d)$ and $\dot{x}_t(s)=\left.\frac{dx(\sigma,x_0,d)}{d\sigma}\right|_{\sigma=t+s}$ for all $s\in[-\tau,0]$. Similarly, $y_t(\theta)=\Lambda_r(\lambda)x(\lambda^\nu(t+\theta),x_0,d)$ and $\dot{y}_t(\theta)=\lambda^\nu\Lambda_r(\lambda)\left.\frac{dx(\sigma,x_0,d)}{d\sigma}\right|_{\sigma=\lambda^\nu(t+\theta)}$ for all $\theta\in[-\lambda^{-\nu}\tau,0]$. Therefore, $x_{\lambda^\nu t}(s)=x(\lambda^\nu t+s,x_0,d)=x(\lambda^\nu(t+\theta),x_0,d)=\Lambda_r^{-1}(\lambda)y_t(\theta)$ and $\dot{x}_{\lambda^\nu t}(s)=\left.\frac{dx(\sigma,x_0,d)}{d\sigma}\right|_{\sigma=\lambda^\nu t+s}=\left.\frac{dx(\sigma,x_0,d)}{d\sigma}\right|_{\sigma=\lambda^\nu(t+\theta)}=\lambda^{-\nu}\Lambda_r^{-1}(\lambda)\dot{y}_t(\theta)$ for all $s\in[-\tau,0]$ with $\theta=\lambda^{-\nu}s$ (i.e. $\theta\in[-\lambda^{-\nu}\tau,0]$). Let us calculate the time derivative of the proposed expression for $y(t,y_0,\delta)$:

$$\begin{aligned}\frac{dy(t,y_0,\delta)}{dt}&=\lambda^\nu\Lambda_r(\lambda)\frac{dx(\lambda^\nu t,x_0,d)}{d\lambda^\nu t}\\&=\lambda^\nu\Lambda_r(\lambda)f(x_{\lambda^\nu t},\dot{x}_{\lambda^\nu t},d(\lambda^\nu t))\\&=f(\Lambda_r(\lambda)x_{\lambda^\nu t},\Lambda_{r+\nu}(\lambda)\dot{x}_{\lambda^\nu t},\Lambda_{\tilde{r}}(\lambda)d(\lambda^\nu t))=f(y_t,\dot{y}_t,\delta(t)).\end{aligned}$$

The obtained equality corresponds to (5).

In [8, 9], using the scaling property of solutions (i.e. an analogue of Proposition 2) it has been shown several interesting characteristics of retarded functional differential equations. For neutral type of time-delay systems those results have the following correspondences:

Corollary 1 *Let the origin be locally asymptotically stable (LAS) for a r-homogeneous system* (3) *with $d=0$ and the degree $\nu=0$, then it is GAS.*

PROOF. Assume that the origin is locally attractive for (3) with a domain of attraction containing the ball B^τ_μ for some $\mu>0$, i.e. for any $\varepsilon>0$ and $x_0\in B^\tau_\mu$ there is $T_{\varepsilon,x_0}\ge0$ such that $|x(t,x_0,0)|_r\le\varepsilon$ for all $t\ge T_{\varepsilon,x_0}$ (by (2) the norms $||\cdot||_{\mathbb{W}}$ and $||\cdot||_r$ can be replaced). For any $\xi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$ there is $x_0\in B^\tau_\mu$, $x_0\ne0$ such that $\xi=\Lambda_r(\lambda)x_0$ for $\lambda=\frac{||\xi||_r}{||x_0||_r}$ and the corresponding unique solution $x(t,\xi,0)=\Lambda_r(\lambda)x(\lambda^\nu t,x_0,0)=\Lambda_r(\lambda)x(t,x_0,0)$ by Proposition 2. Thus, if $x(t,x_0,0)\to0$ for all $x_0\in B^\tau_\mu$ with $t\to+\infty$, then so is $x(t,\xi,0)=\Lambda_r(\lambda)x(t,x_0,0)$ providing global attractiveness and forward completeness.

To prove that local stability implies global, assume that $\sup_{t\ge0}||x(t,x_0,0)||_r\le\sigma(||x_0||_r)$ for all $x_0\in B^\tau_\mu$ and some $\sigma\in\mathcal{K}$. Now take any $\xi\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$, then there is $x_0\in B^\tau_\mu$ with $||x_0||_r=\mu$ such that $\xi=\Lambda_r(\lambda)x_0$ for $\lambda=\mu^{-1}||\xi||_r$ with the corresponding unique solution $x(t,\xi,0)=\Lambda_r(\lambda)x(t,x_0,0)$ by Proposition 2. Therefore,

$$\sup_{t\ge0}||x(t,\xi,0)||_r=\sup_{t\ge0}||\Lambda_r(\lambda)x(t,x_0,0)||_r$$

$$=\lambda\sup_{t\ge0}||x(t,x_0,0)||_r\le||\xi||_r\frac{\sigma(||x_0||_r)}{\mu}=\frac{\sigma(\mu)}{\mu}||\xi||_r,$$

and the system is Lyapunov stable [17, 23, 28].

Corollary 2 *Let* (3) *with $d=0$ be r-homogeneous with degree $\nu\ne0$ and GAS for some delay $0<\tau_0<+\infty$, then it is GAS for any delay $0<\tau<+\infty$ (i.e. DI).*

PROOF. Due to the imposed hypothesis, for the delay $\tau_0>0$ for all $x_0\in\mathbb{W}^{1,\infty}_{[-\tau_0,0]}$ there is a function $\sigma\in\mathcal{K}$ such that $|x(t,x_0,0)|_r\le\sigma(||x_0||_r)$ for all $t\ge0$ and $\lim_{t\to+\infty}|x(t,x_0,0)|_r=0$. Take some $\tau\in(0,+\infty)$ and select an initial condition $y_0\in\mathbb{W}^{1,\infty}_{[-\tau,0]}$, then for $\lambda=\left(\frac{\tau_0}{\tau}\right)^{1/\nu}$ (this λ is well-defined since $\nu\ne0$) there exists $x_0\in\mathbb{W}^{1,\infty}_{[-\tau_0,0]}$ such that $y_0(s)=\Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s\in[-\tau,0]$, and $y(t,y_0,0)=\Lambda_r(\lambda)x(\lambda^\nu t,x_0,0)$ for all $t\ge0$ by Proposition 2. Thus, $\lim_{t\to+\infty}|y(t,y_0,0)|_r=\lambda\lim_{t\to+\infty}|x(\lambda^\nu t,x_0,0)|_r=\lambda\lim_{t\to+\infty}|x(t,x_0,0)|_r=0$ and the solution $y(t,y_0,0)$ is converging asymptotically to the origin. In addition, $|y(t,y_0,0)|_r = \lambda|x(\lambda^\nu t, x_0, 0)|_r \leq \lambda\sigma(||x_0||_r) = \lambda\sigma(\lambda^{-1}||y_0||_r)$ for all $t \geq 0$, which implies stability of the system (3) for the delay τ. The proven convergence to the origin and stability provide the global asymptotic stability of the system for an arbitrary delay $\tau \in (0, +\infty)$.

Corollary 3 *Let the system* (3) *with* $d = 0$ *be* r*-homogeneous with degree* ν *and asymptotically stable with the domain of attraction* B_ρ^τ *for some* $0 < \rho < +\infty$ *for any value of delay* $0 \leq \tau < +\infty$*, then it is DI GAS.*

PROOF. For any $\tau > 0$ take $y_0 \in \mathbb{W}^{1,\infty}_{[-\tau,0]}$, $y_0 \notin B_\rho^\tau$, then according to Proposition 2 there is $0 < \lambda < +\infty$ ($\lambda = \rho^{-1}||y_0||_r$) and $x_0 \in B_\rho^{\lambda^\nu \tau}$ such that $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s \in [-\tau, 0]$ and $y(t,y_0,0) = \Lambda_r(\lambda)x(\lambda^\nu t, x_0, 0)$ for all $t \geq 0$. Since $x(t)$ converges asymptotically to the origin, the same property is satisfied for $y(t)$ and it enters the set B_ρ^τ in a finite time.

Other results presented in [8, 9] for retarded functional differential equations can be similarly extended to (3).

4 Accelerated stabilization of double integrator by measurement of its position

Consider the double integrator system:

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u(t) + d(t),\ y(t) = x_1(t), \qquad (6)$$

where $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$ are the position and velocity, respectively, $u(t) \in \mathbb{R}$ is the control input and $y(t) \in \mathbb{R}$ is the output available for measurements, $d(t) \in \mathbb{R}$ is the disturbance with $d \in \mathcal{L}_\infty$. The goal of this work is to design a static output feedback practically stabilizing the system with an accelerated convergence rate for the case $d = 0$, and ensuring boundedness of the system solutions for $d \neq 0$.

Defining $y(t-h) = x_1(0)$ for $t \in [0, h]$, the control algorithm proposed in this paper is

$$u(t) = -(k_1 + k_2)\lceil y(t) \rfloor^\alpha + k_2 \lceil y(t-h) \rfloor^\alpha,\ t \geq 0, \qquad (7)$$

where $\lceil y \rfloor^\alpha = |y|^\alpha \mathrm{sign}(y)$, $k_1 > 0$ and $k_2 > 0$ are tuning gains, $\alpha > 0$, $\alpha \neq 1$ is a tuning power and $h > 0$ is the delay (if $\alpha = 1$ then the control (7) is linear and it has been studied in [19, 20]).

4.1 Disturbance-free case $d = 0$

Theorem 2 *For* $k_1 > 0$, $k_2 > 0$ *and* $h_0 > 0$*, if the system of linear matrix inequalities*

$$Q \leq 0,\ P > 0,\ q > 0, \qquad (8)$$

$$Q = \begin{bmatrix} Q_{11} & k_2 Z b & Zb \\ k_2 b^\top Z^\top & \left(k_2^2 h^2 - 4\frac{e^{-\varpi h}}{h^2}\right) q & qh^2 k_2 \\ b^\top Z^\top & qh^2 k_2 & qh^2 - \gamma \end{bmatrix},$$

$$Q_{11} = A^\top P + PA + qh^2 A^\top b b^\top A + \varpi P,$$

$$Z = P + qh^2 A^\top,\ A = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 h \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is feasible for some $\varpi > 0$, $\gamma > 0$ *and any* $0 < h \leq h_0$*, then for any* $0 < \eta < +\infty$ *there exists* $\epsilon \in (0, 1)$ *sufficiently small such that* (6), (7) *with* $d = 0$ *is*

a) GAS with respect to the set B_η^h *for any* $\alpha \in (1 - \epsilon, 1)$*;*

b) LAS at the origin from B_η^h *for any* $\alpha \in (1, 1 + \epsilon)$.

Remark 2 *The conditions of the theorem connect the control parameters to be tuned (the gains* $k_1 > 0$ *and* $k_2 > 0$*, the maximal admissible delay* $h_0 > 0$*, the nonlinear power* $\alpha > 0$*) and the variables of linear matrix inequalities (*$\varpi > 0$, $\gamma > 0$, $Q \leq 0$, $P > 0$ *and* $q > 0$*), which are obtained solving* (8) *(a grid on the interval* $(0, h_0]$ *can be used). It is worth noting that* (8) *is always satisfied for sufficiently small values of* h *[19, 20]. The bound* $\epsilon > 0$ *on admissible deviation of* α *from* 1 *(deviation of* (7) *from a linear control) or the estimate on the domain of attraction* $\eta > 0$ *are derived as the functions of the above variables and parameters in the proof. In applications, a choice of* α *is also possible based on the error and trial that reduces the numeric conservatism.*

PROOF. A verification shows that for any $\alpha \geq 0$ the system (6), (7) with $d = 0$ is r-homogeneous for $r_1 = 1$ and $r_2 = \frac{\alpha+1}{2}$ of degree $\nu = \frac{\alpha-1}{2}$.

Following the ideas of [19, 20] let us rewrite the system (6), (7) using the linear feedback with $\alpha = 1$ and a nonlinear error δ:

$$\dot{x}_2(t) = -(k_1 + k_2)y(t) + k_2 y(t-h) + \delta(t), \qquad (9)$$

where $\delta(t) = (k_1 + k_2)(y(t) - \lceil y(t) \rfloor^\alpha) + k_2(\lceil y(t-h) \rfloor^\alpha - y(t-h)) + d(t)$. Note that

$$y(t-h) = y(t) - h\dot{y}(t) + R(t),\ R(t) = \int_{t-h}^{t} (s - t + h)\ddot{y}(s)ds,$$

then (9) can be represented as follows:

$$\dot{x}(t) = Ax(t) + b\left(k_2 R(t) + \delta(t)\right) \qquad (10)$$

for $x(t) = [x_1(t) \quad x_2(t)]^\top$, which is in the form (3) and $x_t \in \mathbb{W}^{1,\infty}_{[-h,0]}$ is the state. In other words, the dynamics in (10) is neutral since $R(t)$ is a function of delayed values of $\dot{x}_2(s)$ with $s \in [t-h, t]$, then in order to analyze ISS property of (10) with respect to the input $\delta(t)$ let us consider an ISS LKF:

$$V(x_t, \dot{x}_t) = x^\top(t)Px(t) + qW(x_t, \dot{x}_t),$$
$$W(x_t, \dot{x}_t) = \int_{t-h}^{t} e^{\varpi(s-t)}(s-t+h)^2 \dot{x}_2^2(s)ds,$$

where $P = P^\top > 0$, $\varpi > 0$ and $q > 0$ are solutions of (8), and $x_t \in \mathbb{W}^{1,\infty}_{[-h,0]}$. For $W(t) = W(x_t, \dot{x}_t)$ a direct computation gives:

$$\dot{W}(t) = h^2\dot{x}_2^2(t) - 2\int_{t-h}^{t} e^{\varpi(s-t)}(s-t+h)\dot{x}_2^2(s)ds - \varpi W(t),$$

and applying Jensen's inequality [18] we obtain:

$$\int_{t-h}^{t} e^{\varpi(s-t)}(s-t+h)\dot{x}_2^2(s)ds \geq 2\frac{e^{-\varpi h}}{h^2}R^2(t),$$
$$\dot{W}(t) \leq h^2\dot{x}_2^2(t) - 4\frac{e^{-\varpi h}}{h^2}R^2(t) - \varpi W(t).$$

Since $V(t) = V(x_t, \dot{x}_t)$ is an absolutely continuous function, the full derivative of $V(t)$ for (10) can now be estimated as follows:

$$\begin{aligned}\dot{V}(t) &\leq x^\top(t)[A^\top P + PA]x(t) + 2x^\top(t)Pb(k_2R(t) + \delta(t)) \\ &+q\left[h^2\dot{x}^\top(t)bb^\top\dot{x}(t) - 4\frac{e^{-\varpi h}}{h^2}R^2(t) - \varpi W(t)\right] \\ &= [x^\top(t)\ R^\top(t)\ \delta^\top(t)]Q[x^\top(t)\ R^\top(t)\ \delta^\top(t)]^\top \\ &-\varpi\left(x^\top(t)Px(t) + qW(t)\right) + \gamma\delta^2(t).\end{aligned}$$

Since $Q \leq 0$, finally we obtain

$$\dot{V}(t) \leq -\varpi V(t) + \gamma\delta^2(t), \tag{11}$$

which implies that V is an ISS LKF for (10) and the system possesses ISS property with respect to the input δ due to Theorem 1 provided that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x(t)|) \leq V(x_t, \dot{x}_t) \leq \alpha_2(\|x_t\|_{\mathbb{W}})$ for all $x_t \in \mathbb{W}^{1,\infty}_{[-h,0]}$. Obviously, $\alpha_1(s) = \lambda_{\min}(P)s^2$ and in order to evaluate α_2 let us consider

$$\begin{aligned}W(x_t, \dot{x}_t) &\leq \int_{t-h}^{t}(s-t+h)^2\dot{x}_2^2(s)ds \leq h^2\int_{t-h}^{t}\dot{x}_2^2(s)ds \\ &\leq h^3 \sup_{s\in[t-h,t]}\dot{x}_2^2(s) \leq h^3\|\dot{x}_t\|^2 \leq h^3\|x_t\|^2_{\mathbb{W}},\end{aligned}$$

then $\alpha_2(s) = \lambda_{\max}(P)s^2 + qh^3s^2$.

Let $d = 0$, then $\delta(t) = (k_1 + k_2)(x_1(t) - \lceil x_1(t)\rfloor^\alpha) + k_2(\lceil x_1(t-h)\rfloor^\alpha - x_1(t-h))$ and from the performed analysis there exists $\varepsilon > 0$ such that $\inf_{x_t\in\Omega} V(x_t, \dot{x}_t) \geq \varepsilon$, where $\Omega = B_1^h \setminus B_{0.5}^h$. We have $|x_1(t)| \leq 1$ and $|x_1(t-h)| \leq 1$ for $x_t \in \Omega$. Since $\delta^2(t) \leq 2(k_1+k_2)^2(|x_1(t)| - |x_1(t)|^\alpha)^2 + 2k_2^2(|x_1(t-h)|^\alpha - |x_1(t-h)|)^2$ and due to the fact that the function $g(s) = (s - s^\alpha)^2$ for $s \in [0,1]$ and $\alpha > 0$, $\alpha \neq 1$ admits an estimate $g(s) \leq (1-\alpha)^2\rho^2(\alpha)$ with $\rho(\alpha) = \alpha^{\frac{\alpha}{1-\alpha}}$, we obtain

$$\dot{V} \leq -\varpi\varepsilon + 2\gamma[(k_1+k_2)^2 + k_2^2](1-\alpha)^2\rho^2(\alpha)$$

for all $x_t \in \Omega$ (a similar analysis can be made for $s \geq 1$). Note that $\rho(0) = 1$ and $\frac{d}{d\alpha}\rho(\alpha) < 0$, then always there exists $\epsilon \in (0,1)$ such that

$$4\gamma[(k_1+k_2)^2 + k_2^2](1-\alpha)^2\rho^2(\alpha) \leq \varpi\varepsilon \quad \forall\alpha \in [1-\epsilon, 1+\epsilon]$$

and $\dot{V} \leq -0.5\varpi\varepsilon \leq 0$ for all $x_t \in \Omega$ and $0 < h \leq h_0$ with $d = 0$. Thus, there exists $0.5 \leq \eta_1 < \eta_2 \leq 1$ such that all trajectories of (6), (7) initiated on the sphere $\mathcal{S}^h_{\eta_2}$ reach the set $B^h_{\eta_1}$ for any $0 < h \leq h_0$ with the selected $k_1 > 0$, $k_2 > 0$ and $\alpha \in [1-\epsilon, 1+\epsilon]$. The stability follows the properties of the functional V.

Since the system (6), (7) with $d = 0$ is homogeneous for $r = [1\ \frac{\alpha+1}{2}]^\top$, then according to Proposition 2 its solutions are interrelated via dilation of initial conditions and scaling of delay. First, consider the case $\alpha \in (0,1)$, then the system is homogeneous of negative degree $\nu = \frac{\alpha-1}{2}$. Select some initial condition $\xi_0 \in \mathbb{W}^{1,\infty}_{[-h,0]} \setminus B^h_{\eta_2}$, define $\lambda = \eta_2^{-1}\|\xi_0\|_r > 1$ and take $x_0 \in \mathcal{S}^{\lambda^\nu h}_{\eta_2}$ such that $\xi_0(s) = \Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s \in [-h, 0]$. Since $\lambda^\nu h < h$, then for any $0 < h \leq h_0$ the trajectory initiated at x_0 reaches $B^{\lambda^\nu h}_{\eta_1}$. Consequently, the trajectory starting at ξ_0 enters into $B^h_{\lambda\eta_1}$. Repeating these arguments it is possible to prove that the set $B^h_{\eta_1}$ is GAS for any $0 < h \leq h_0$. Second, consider the case $\alpha > 1$, then the system is homogeneous of positive degree $\nu = \frac{\alpha-1}{2}$. Select some initial condition $\xi_0 \in B^h_{\eta_1}$, define $\lambda = \eta_2^{-1}\|\xi_0\|_r < 1$ and take $x_0 \in \mathcal{S}^{\lambda^\nu h}_{\eta_2}$ such that $\xi_0(s) = \Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s \in [-h, 0]$. Since $\lambda^\nu h < h$, then for any $0 < h \leq h_0$ the trajectory initiated at x_0 reaches the ball $B^{\lambda^\nu h}_{\eta_1}$, while the trajectory corresponding to ξ_0 enters into $B^h_{\lambda\eta_1}$. Repeating these arguments it is possible to prove that the origin is LAS for any $0 < h \leq h_0$ from the set of initial conditions $B^h_{\eta_2}$.

Since for $\alpha = 1$ the origin is globally attracting, then selecting $\epsilon \in (0,1)$ sufficiently small it is possible to assign arbitrary values for the levels $0 < \eta_1 < \eta_2 < +\infty$.

Remark 3 *The state of the original system* (6) *is defined in* $\mathbb{R}^2$. *After introduction of the control law* (7)*, the state* x_t *of the closed-loop system becomes from* $C^2_{[-h,0]}$. *And finally, since for the stability analysis we use an auxiliary representation* (10) *(a linear approximation of* (6), (7)*), with* R *that depends on* $\dot{x}_t$*, and the corresponding ISS LKF* $V(x_t, \dot{x}_t)$*, then the related stability results are obtained in the Sobolev space* $\mathbb{W}^{1,\infty}_{[-h,0]}$. *These stability properties are only local for the original system* (6), (7)*, and we use the theory of homogeneity (whose extension to the class of neutral systems and the space* $\mathbb{W}^{1,\infty}_{[-h,0]}$ *is given in Section 3) to improve the estimates on the domain of stability and attraction.*

The requirement that the matrix inequalities (8) have to be verified for any $0 < h \leq h_0$ may be restrictive for given gains k_1 and k_2, then another local result can be obtained by relaxing this constraint:

Corollary 4 *For any $k_1 > 0$, $k_2 > 0$ and $0 < h_1 < h_0$, let the system of linear matrix inequalities (8) be verified for some $\varpi > 0$, $\gamma > 0$ and all $h_1 \le h \le h_0$. Then for any $0 < \rho_1 < +\infty$ there exist $\epsilon \in (0,1)$ sufficiently small and $\rho_2 > \rho_1$ such that (6), (7) with $d = 0$ is asymptotically stable with respect to the set $B^h_{\rho_1}$ with the region of attraction $B^h_{\rho_2}$ for any $\alpha \in (1-\epsilon, 1+\epsilon)$.*

PROOF. The result follows the arguments of the proof of Theorem 2 taking into account that if $h_1 \le h \le h_0$ (and not $0 < h \le h_0$), then dilating the initial conditions to infinity for $\alpha \in (1-\epsilon, 1)$ or in a vicinity of the origin for $\alpha \in (1, 1+\epsilon)$ is limited by $\lambda = (h_1/h_0)^{1/\nu}$.

Note that with $d = 0$ in all cases, for $\nu \neq 0$, the global stability at the origin cannot be obtained in (6), (7) (due to homogeneity of the system, following the result of Corollary 2, the globality implies DI result), while in the linear case with $\nu = 0$ such a result is possible to derive for any $0 < h \le h_0$. Then it is necessary to justify a need in the nonlinear control with $\nu \neq 0$ comparing to the linear feedback with the same gains. An answer to this question is presented in the following result, and to this end denote for the system (6), (7):

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) = \arg\inf_{T \ge h} \sup_{t \ge T} \sup_{x_0 \in \mathcal{S}^h_{\rho_2}} |x(t, x_0, 0)|_r \le \rho_1$$

as the time of convergence of all trajectories initiated on the sphere $\mathcal{S}^h_{\rho_2}$ to the set $B^h_{\rho_1}$ provided that the delay h and the power α are applied in the feedback (with $d = 0$).

Proposition 1 *For given $k_1 > 0$, $k_2 > 0$, $h_0 > 0$, let the linear matrix inequalities (8) be verified for some $\varpi > 0$, $\gamma > 0$ and any $0 < h \le h_0$. Then there exist $\epsilon \in (0,1)$ sufficiently small and $0 < \rho_1 \le 0.5$, $2 \le \rho_2 < +\infty$ such that in (6), (7) with $d = 0$:*

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) < \mathcal{T}(1, \rho_1, \rho_2, h) \qquad (12)$$

for any $\alpha \in (1-\epsilon, 1)$ or $\alpha \in (1, 1+\epsilon)$ and all

$$h \in \begin{cases} [\rho_2^{-\nu} h_1, 2^{-\nu}\rho_1^{-\nu} h_0] & \alpha \in (1-\epsilon, 1) \\ [2^{-\nu}\rho_1^{-\nu} h_1, \rho_2^{-\nu} h_0] & \alpha \in (1, 1+\epsilon) \end{cases},$$

where $\nu = \frac{\alpha-1}{2}$ is the degree of homogeneity and $0 < h_1 \le (2\rho_1\rho_2^{-1})^{|\nu|} h_0$.

In other words, this result claims that for any feedback gains k_1 and k_2, if the conditions of Theorem 2 are satisfied, then the nonlinear system (6), (7) with $\nu \neq 0$ ($\alpha \neq 1$) is always converging faster than its linear analog with $\nu = 0$ ($\alpha = 1$) between properly selected levels ρ_1 and ρ_2 (which values depend on smaller or higher than 1 is α) for an appropriately selected delay h. A procedure for selection of all these values is given in the proof below.

PROOF. Since all conditions of Theorem 2 are validated, then for an arbitrary $\eta > 0$ there is $\epsilon \in (0,1)$ such that for any $0 < h \le h_0$ the system (6), (7) with $d = 0$ is GAS with respect to the set B^h_η for any $\alpha \in (1-\epsilon, 1)$ or LAS at the origin from B^h_η for any $\alpha \in (1, 1+\epsilon)$. For $\alpha = 1$ the system is GAS at the origin for all $0 < h \le h_0$. Therefore, for a properly adjusted ϵ there exists some $T_\epsilon \in \mathbb{R}_+$ such that

$$\sup_{h_1 < h \le h_0} \sup_{\alpha \in (1-\epsilon, 1+\epsilon)} \mathcal{T}(\alpha, 0.5, 1, h) \le T_\epsilon,$$

and by homogeneity, the time of convergence to $B^{\lambda^{-\nu}h}_{0.5\lambda}$ from $\mathcal{S}^{\lambda^{-\nu}h}_{\lambda}$ with $\lambda \in \mathbb{R}_+$ is upper bounded by $\lambda^{-\nu} T_\epsilon$ for all $h \in (h_1, h_0]$ and $\alpha \in (1-\epsilon, 1+\epsilon)$.

Consider the case $\alpha \in (1-\epsilon, 1)$ and, consequently, $\nu < 0$. Select $\rho_1 = 2^{-z} < 1$ and $\rho_2 = 2^Z > 1$ for some integers $z \ge 1$ and $Z \ge 1$, then

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) \le T_\epsilon \left(\sum_{i=0}^{z-1} 2^{\nu i} + \sum_{i=1}^{Z} 2^{-\nu i} \right)$$

for all $h \in [2^{-\nu Z} h_1, 2^{\nu(z-1)} h_0] = [\rho_2^{-\nu} h_1, 2^{-\nu}\rho_1^{-\nu} h_0]$, and the latter interval is not empty due to the selection of $h_1 \le (2\rho_1\rho_2^{-1})^{-\nu} h_0$. Since $\sum_{i=0}^{+\infty} 2^{\nu i} = \frac{1}{1-2^\nu}$ and $\sum_{i=0}^{Z-1} 2^{-\nu i} = \frac{1-2^{-\nu Z}}{1-2^{-\nu}}$, then the time $\mathcal{T}(\alpha, \rho_1, \rho_2, h)$ is bounded for any growing z, and Z has to be finite:

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) \le T_\epsilon \left(\frac{1 - 2^{\nu z}}{1 - 2^\nu} + 2^{-\nu} \frac{1 - 2^{-\nu Z}}{1 - 2^{-\nu}} \right) = T_\epsilon \frac{\rho_2^{-\nu} - \rho_1^{-\nu}}{1 - 2^\nu}.$$

Similarly, for the case $\alpha \in (1, 1+\epsilon)$ and $\nu > 0$, select $\rho_2 = 2^z > 1$ and $\rho_1 = 2^{-Z} < 1$ for integers $z \ge 1$ and $Z \ge 1$, then the estimate

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) \le T_\epsilon \left(\sum_{i=1}^{z} 2^{-\nu i} + \sum_{i=0}^{Z-1} 2^{\nu i} \right)$$

is satisfied for $h \in [2^{\nu(Z-1)} h_1, 2^{-\nu z} h_0] = [2^{-\nu}\rho_1^{-\nu} h_1, \rho_2^{-\nu} h_0]$, and the latter interval is non-empty for $h_1 \le (2\rho_1\rho_2^{-1})^\nu h_0$. Again, $\sum_{i=0}^{+\infty} 2^{-\nu i} = \frac{1}{1-2^{-\nu}}$ and $\sum_{i=0}^{Z-1} 2^{\nu i} = \frac{1-2^{\nu Z}}{1-2^\nu}$. Therefore, similarly as for the case $\nu < 0$, the value of z can be selected arbitrary and Z should be limited:

$$\mathcal{T}(\alpha, \rho_1, \rho_2, h) \le T_\epsilon \left(2^{-\nu} \frac{1 - 2^{-\nu z}}{1 - 2^{-\nu}} + \frac{1 - 2^{\nu Z}}{1 - 2^\nu} \right) = T_\epsilon \frac{\rho_2^{-\nu} - \rho_1^{-\nu}}{1 - 2^\nu}.$$

For the case $\alpha = 1$ (and $\nu = 0$) the time of convergence is unbounded with growing $z \ge 1$ or $Z \ge 1$ (a well-known property of the exponentially stable systems):

$$\mathcal{T}(1, \rho_1, \rho_2, h) \ge T_h \sum_{i=1}^{z+Z} 1 = (z+Z) T_h = T_h (\log_2 \rho_2 - \log_2 \rho_1),$$

where T_h is the fastest time of transition between the levels ρ_1 and ρ_2 for $\nu = 0$. Thus, (12) is verified if

$$T_\epsilon \frac{\rho_2^{-\nu} - \rho_1^{-\nu}}{1 - 2^\nu} < T_h (\log_2 \rho_2 - \log_2 \rho_1),$$

which for any $\nu < 0$ ($\nu > 0$), T_ϵ, h, T_h and ρ_2 (ρ_1) can be guaranteed by decreasing ρ_1 (increasing ρ_2). Therefore, in both cases, for $\alpha \in (1-\epsilon, 1)$ and $\alpha \in (1, 1+\epsilon)$, there exist

$0 < \rho_1 < \rho_2 < +\infty$ such that the relation (12) is satisfied for some h in the given intervals.

The result of Proposition 1 provides a motivation for using nonlinear control in this setting: playing with degree of homogeneity of the closed-loop system it is possible to accelerate the obtained linear feedback.

Remark 4 *Note that another, conventional acceleration solution, which consists in increasing k_1 and k_2, may be unfeasible for given delay h_0 (i.e. (8) may loose validity for higher control gains and given delay).*

Remark 5 *Since the time T_ϵ is evaluated on a bounded interval of delays $[h_1, h_0]$, then the result of Proposition 1 can also be used in the conditions of Corollary 4.*

4.2 Robust stability analysis

Now we extend the result of Theorem 2 for $d \neq 0$.

Theorem 3 *For $k_1 > 0$, $k_2 > 0$ and $h_0 > 0$, if the linear matrix inequalities (8) are feasible for some $\varpi > 0$, $\gamma > 0$ and any $0 < h \leq h_0$, then there exists $\epsilon \in (0, 1)$ sufficiently small such that the system (6), (7) has*

a) practical AG and GS properties for any $\alpha \in (1 - \epsilon, 1)$;

b) (X, D)-AG and (X, D)-LS properties for any $\alpha \in (1, 1+\epsilon)$, for some $X \in \mathbb{R}_+$ and $D \in \mathbb{R}_+$.

The values of X and D are evaluated in the proof below.

PROOF. Repeating the arguments presented in the proof of Theorem 2, the system (6), (7) admits ISS property with respect to the auxiliary input δ, and the inequality (11) is satisfied. Consider the set $\Omega = B_1^h \setminus B_{0.5}^h$ ($|x_1(t)| \leq 1$ and $|x_1(t-h)| \leq 1$ for $x_t \in \Omega$), then

$$\delta^2(t) \leq 3(k_1 + k_2)^2 \left(|x_1(t)| - |x_1(t)|^\alpha\right)^2 + 3k_2^2 \left(|x_1(t-h)|^\alpha - |x_1(t-h)|\right)^2 + 3d^2(t)$$

and due to the fact that the function $g(s) = (s - s^\alpha)^2$ for $s \in [0, 1]$ and $\alpha > 0$ admits an estimate

$$g(s) \leq (1-\alpha)^2 \rho^2(\alpha), \quad \rho(\alpha) = \alpha^{\frac{\alpha}{1-\alpha}},$$

we obtain

$$\dot{V} \leq -\varpi V + 3\gamma[(k_1 + k_2)^2 + k_2^2](1-\alpha)^2\rho^2(\alpha) + 3||d||_\infty^2$$

for all $x_t \in \Omega$. As in the proof of Theorem 2, there exists $\varepsilon > 0$ such that $\inf_{x_t \in \Omega} V(x_t, \dot{x}_t) \geq \varepsilon$. Assume that

$$V(x_t, \dot{x}_t) \geq \frac{9}{\varpi}||d||_\infty^2, \tag{13}$$

i.e. $\chi(s) = \frac{9}{\varpi}s^2$ in Definition 5, then

$$\dot{V} \leq -\frac{2}{3}\varpi V + 3\gamma[(k_1 + k_2)^2 + k_2^2](1-\alpha)^2\rho^2(\alpha)$$

Note that $\rho(0) = 1$ and $\frac{d}{d\alpha}\rho(\alpha) < 0$, then always there exists $\epsilon \in (0, 1)$ such that

$$9\gamma[(k_1 + k_2)^2 + k_2^2](1-\alpha)^2\rho^2(\alpha) \leq \varpi\varepsilon \quad \forall \alpha \in [1-\epsilon, 1+\epsilon],$$

which results in $\dot{V} \leq -\frac{1}{3}\varpi\varepsilon$ for all $x_t \in \Omega$ and $0 < h \leq h_0$. Consequently, there exists $0.5 \leq \eta_1 < \eta_2 \leq 1$ such that all trajectories of (6), (7) initiated on $\mathcal{S}_{\eta_2}^h$ reach $B_{\eta_1}^h$ for any $0 < h \leq h_0$ with the selected $k_1 > 0$, $k_2 > 0$ and $\alpha \in [1-\epsilon, 1+\epsilon]$ provided that (13) holds. The last observation implies that $||d||_\infty \leq \frac{1}{3}\sqrt{\varpi\varepsilon}$. In addition, $V(x_t, \dot{x}_t) \leq \alpha_2 \circ \overline{\rho}_r(\eta_2)$ for all $x_t \in \Omega$, where $\overline{\rho}_r$ is given in (2) and $\alpha_2(s) = h^3 s^2$ following the proof of Theorem 2. Therefore,

$$\lim_{t \to +\infty} |x(t, x_0, d)| \leq \max\{\overline{\rho}_r(\eta_1), \alpha_1^{-1}(\frac{9}{\varpi}||d||_\infty^2)\}$$

$$= \max\{\overline{\rho}_r(\eta_1), 3\sqrt{\frac{\lambda_{\min}^{-1}(P)}{\varpi}}||d||_\infty\},$$

$$|x(t, x_0, d)| \leq \alpha_1^{-1} \circ \alpha_2 \circ \overline{\rho}_r(\eta_2) = \sqrt{\lambda_{\min}^{-1}(P)h^3}\overline{\rho}_r(\eta_2)$$

for all $t \geq 0$, for all $x_0 \in \mathcal{S}_{\eta_2}^h$ and $||d||_\infty \leq \frac{\sqrt{\varpi\varepsilon}}{3}$, where we used the fact established in the proof of Theorem 2 that $\alpha_1(s) = \lambda_{\min}(P)s^2$.

The system (6), (7) with $d = 0$ is homogeneous for $r = [1 \ \frac{\alpha+1}{2}]^\top$ of degree $\nu = \frac{\alpha-1}{2}$, and for $d \neq 0$ the property (4) is also satisfied with $\tilde{r} = \alpha$. Then according to Proposition 2 the solutions of (6), (7) are interrelated via dilation of initial conditions, delay and disturbance.

For $\alpha \in (1-\epsilon, 1)$ and a negative degree ν, select some initial condition $\xi_0 \in \mathbb{W}_{[-h,0]}^{1,\infty} \setminus B_{\eta_2}^h$. Define $\lambda = \eta_2^{-1}\|\xi_0\|_r > 1$ and take $x_0 \in \mathcal{S}_{\eta_2}^{\lambda^\nu h}$ such that $\xi_0(s) = \Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s \in [-h, 0]$. Consider an input $\Delta \in \mathcal{L}_\infty$ for the initial condition ξ_0, then $\Delta(t) = \lambda^\alpha d(\lambda^\nu t)$ for all $t \geq 0$, where $d \in \mathcal{L}_\infty$ is an input for the initial condition x_0. Since $\lambda^\nu h < h$, then for any $0 < h \leq h_0$ the trajectory initiated at x_0 reaches $B_{\eta_1}^{\lambda^\nu h}$ provided that (13) is valid for this d. Consequently,

$$\lim_{t \to +\infty} |x(t, \xi_0, \Delta)| \leq \begin{cases} \lambda^{r_{\max}} & \lambda \geq 1 \\ \lambda^{r_{\min}} & \lambda < 1 \end{cases} \lim_{t \to +\infty} |x(\lambda^\nu t, x_0, d)|$$

$$\leq \begin{cases} \lambda^{r_{\max}} & \lambda \geq 1 \\ \lambda^{r_{\min}} & \lambda < 1 \end{cases} \max\{\overline{\rho}_r(\eta_1), 3\sqrt{\frac{\lambda_{\min}^{-1}(P)}{\varpi}}\lambda^{-\alpha}||\Delta||_\infty\}$$

$$\leq \lambda \max\{\overline{\rho}_r(\eta_1), 3\sqrt{\frac{\lambda_{\min}^{-1}(P)}{\varpi}}\lambda^{-\alpha}||\Delta||_\infty\},$$

$$|x(t, \xi_0, \Delta)| \leq \begin{cases} \lambda^{r_{\max}} & \lambda \geq 1 \\ \lambda^{r_{\min}} & \lambda < 1 \end{cases} |x(\lambda^\nu t, x_0, d)|$$

$$\leq \begin{cases} \lambda^{r_{\max}} & \lambda \geq 1 \\ \lambda^{r_{\min}} & \lambda < 1 \end{cases} \sqrt{\lambda_{\min}^{-1}(P)h^3}\overline{\rho}_r(\eta_2) \quad \forall t \geq 0$$

$$\leq \lambda\sqrt{\lambda_{\min}^{-1}(P)h^3}\overline{\rho}_r(\eta_2) \quad \forall t \geq 0$$

and $||\Delta||_\infty \leq \lambda^\alpha \frac{\sqrt{\varpi\varepsilon}}{3}$. Recall that $\lambda = \eta_2^{-1}\|\xi_0\|_r > 1$, then repeating the same argumentation (the trajectory initiated at ξ_0 enters into $B_{\lambda\eta_1}^h$ for any $0 < h \leq h_0$ provided that the input restriction is respected) we conclude

$$\lim_{t\to+\infty}|x(t,\xi_0,\Delta)|\le\chi_1,\ |x(t,\xi_0,\Delta)|\le\chi_2\max\{\|\xi_0\|_r,\eta_2\}\ \forall t\ge 0$$

for all $\xi_0\in\mathbb{W}^{1,\infty}_{[-h,0]}$ provided that $||\Delta||_\infty\le\|x(t,\xi_0,\Delta)\|_r^\alpha\chi_3$, where $\chi_1=\max\{\overline{\rho}_r(\eta_1),\sqrt{\lambda^{-1}_{\min}(P)\varepsilon}\}$, $\chi_2=\eta_2^{-1}\sqrt{\lambda^{-1}_{\min}(P)h^3\overline{\rho}_r(\eta_2)}$ and $\chi_3=\frac{\sqrt{\varpi\varepsilon}}{3}$. Hence, it is possible to establish the practical AG and GS properties for the system (6), (7):

$$\lim_{t\to+\infty}|x(t,\xi_0,\Delta)|\le\max\{\chi_1,\overline{\rho}_r(\sqrt[\alpha]{\chi_3^{-1}||\Delta||_\infty})\},$$

$$|x(t,\xi_0,\Delta)|\le\max\{\chi_2\max\{\|\xi_0\|_r,\eta_2\},\overline{\rho}_r(\sqrt[\alpha]{\chi_3^{-1}||\Delta||_\infty})\}$$

for all $t\ge0$, for all $\xi_0\in\mathbb{W}^{1,\infty}_{[-h,0]}$ and all $\Delta\in\mathcal{L}_\infty$.

For $\alpha\in(1,1+\epsilon)$ and a positive degree ν, chose some initial condition $\xi_0\in B^h_{\eta_1}$, define $\lambda=\eta_2^{-1}\|\xi_0\|_r<1$ and take $x_0\in\mathcal{S}^{\lambda^\nu h}_{\eta_2}$ such that $\xi_0(s)=\Lambda_r(\lambda)x_0(\lambda^\nu s)$ for $s\in[-h,0]$. Consider an input $\Delta\in\mathcal{L}_\infty$ for the initial condition ξ_0, then $\Delta(t)=\lambda^\alpha d(\lambda^\nu t)$ for all $t\ge0$, where $d\in\mathcal{L}_\infty$ with $||d||_\infty\le\frac{\sqrt{\varpi\varepsilon}}{3}$ is an input for the initial condition x_0. Since $\lambda^\nu h<h$, then as before we obtain:

$$\lim_{t\to+\infty}|x(t,\xi_0,\Delta)|\le\lambda\max\{\overline{\rho}_r(\eta_1),3\sqrt{\frac{\lambda^{-1}_{\min}(P)}{\varpi}}\lambda^{-\alpha}||\Delta||_\infty\},$$

$$|x(t,\xi_0,\Delta)|\le\lambda\sqrt{\lambda^{-1}_{\min}(P)h^3\overline{\rho}_r(\eta_2)}\quad\forall t\ge0,\ ||\Delta||_\infty\le\lambda^\alpha\frac{\sqrt{\varpi\varepsilon}}{3}.$$

In this case $\lambda=\eta_2^{-1}\|\xi_0\|_r<1$, then repeating the argumentation we conclude that

$$\lim_{t\to+\infty}|x(t,\xi_0,\Delta)|\le0,\ |x(t,\xi_0,\Delta)|\le\chi_2\|\xi_0\|_r\quad\forall t\ge0$$

for all $\xi_0\in B^h_{\eta_2}$ for any $0<h\le h_0$ provided that

$$||\Delta||_\infty\le\|x(t,\xi_0,\Delta)\|_r^\alpha\chi_3.$$

Consequently, $(\underline{\rho}_r^{-1}(\eta_2),\frac{\sqrt{\varpi\varepsilon}}{3})$-AG and $(\underline{\rho}_r^{-1}(\eta_2),\frac{\sqrt{\varpi\varepsilon}}{3})$-LS properties are substantiated for the system (6), (7).

Let us consider results of application of the proposed control and an illustration of the obtained acceleration.

5 Example

Take $h_0=0.2$,$k_1=0.5$, $k_2=0.1$, $\varpi=10^{-6}$ and $q=5.1\times10^{-12}$, then for

$$P=10^{-8}\times\begin{bmatrix}0.1&0.00\\0.00&0.2\end{bmatrix},\ \gamma=2.81\times10^{-2},$$

the matrix inequalities (8) are satisfied for $h_1<h\le h_0$ with $h_1=5\times10^{-4}$, and the results of verification are presented in Fig. 1. Thus, all conditions of Corollary 4 are verified. In the disturbance-free case, the errors of regulation obtained in simulation of the system (6), (7) with delay h_0 for different initial conditions with $\alpha=0.9$ and $\alpha=1.1$, in comparison with the linear controller with $\alpha=1$, are shown in Fig. 2 (the solid lines represent the trajectories of the system with $\alpha\neq1$ and the dashed ones correspond to $\alpha=1$, since the plots are given in a logarithmic scale, then the latter trajectories are close to straight lines). As we can conclude, in the nonlinear case the convergence is much faster than in the linear one close to the origin for $\alpha\in(0,1)$ and far outside for $\alpha>1$, which confirms the statement of Proposition 1. Note that the value of η (the radius of the set to which the trajectories converge for $\alpha<1$ or from which they converge to the origin for $\alpha>1$) is not restrictive. The results of the system simulation with $\alpha=0.9$ and $d(t)=0.001\sin(0.1t)$ are presented in Fig. 3, and as we can conclude all trajectories converge to a vicinity of the origin proportional to the amplitude of the disturbance and faster than in the linear case.

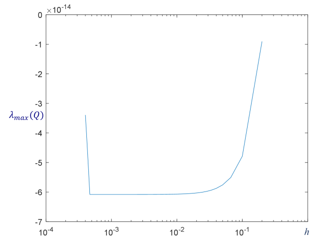
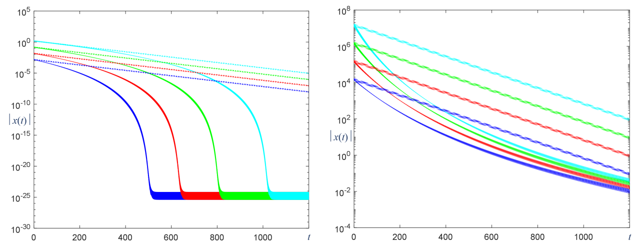


Fig. 1. The results of verification of (8) for different h



(a) $\alpha=0.9$ (b) $\alpha=1.1$

Fig. 2. Trajectories of stabilized double integrator with $d=0$

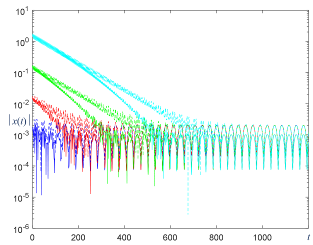


Fig. 3. Trajectories with $\alpha=0.9$ and $d\neq0$

6 Conclusions

The theory of homogeneity is extended to neutral type systems with state space $\mathbb{W}^{1,\infty}_{[-h,0]}$. The new notion of homogeneity is further applied to static output-feedback stabilization of a second order dynamics using a nonlinear delayed control law that achieves accelerated rates of convergence comparatively to the related linear regulator. The control does not need an estimation of velocities, and the applicability of the approach can be checked by resolving linear matrix inequalities. Robustness with respect to additive matched perturbations is assessed. The efficiency of the proposed approach is demonstrated in simulations and comparison with a linear controller.

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